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The Yang-Baxter equation, symmetric functions, and Schubert polynomials

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Abstract

We present an approach to the theory of Schubert polynomials, corresponding symmetric functions, and their generalizations that is based on exponential solutions of the Yang–Baxter equation. In the case of the solution related to the nilCoxeter algebra of the symmetric group, we recover the Schubert polynomials of Lascoux and Schützenberger, and provide simplified proofs of their basic properties, along with various generalizations thereof. Our techniques make use of an explicit combinatorial interpretation of these polynomials in terms of configurations of labelled pseudo-lines.

Keywords: Yang–Baxter equation; Schubert polynomials; Symmetric functions

1. Introduction

The Yang–Baxter operators $h_i(x)$ satisfy the following relations (cf. [1,7]):

$$h_i(x)h_j(y) = h_j(y)h_i(x) \quad \text{if } |i-j| \geqslant 2;$$

$$h_i(x)h_{i+1}(x+y)h_i(y) = h_{i+1}(y)h_i(x+y)h_{i+1}(x);$$

The role the representations of the Yang–Baxter algebra play in the theory of quantum groups [9], the theory of exactly solvable models in statistical mechanics [1], low-dimensional topology [7,27,16], the theory of special functions, and other branches of mathematics (see, e.g., the survey [5]) is well known.

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We study the connections between the Yang–Baxter algebra and the theory of symmetric functions and Schubert polynomials. Let us add to the above conditions the equation

$$h_i(x)h_i(y) = h_i(x+y),$$

thus getting the so-called colored braid relations (see [17,14] for examples of their representations). It turns out that, once these relations hold, one can introduce a whole class of symmetric functions (and even 'double', or 'super-' symmetric functions) and respective analogues of the [double] Schubert polynomials [22,25] as well. These analogues are proved to have many of the properties of their prototypes; e.g., we generalize the Cauchy identities and the principal specialization formula.

The simplest solution of the above equations involves the nilCoxeter algebra of the symmetric group [14]. Exploring this special case, we construct super-analogues of Stanley's symmetric functions G_w (see [29]), provide another combinatorial interpretation of Schubert polynomials $\mathfrak{S}_w$ of Lascoux and Schützenberger, and reprove the basic facts concerning G_w's and $\mathfrak{S}_w$'s. Recently, the construction of this paper has been used [2] to produce a Pieri rule for Schubert polynomials and yet another algorithm that generates the monomials of $\mathfrak{S}_w$.

Other solutions of the main relations are also given. One of them involves Hecke algebras, another one the universal enveloping algebra of the Lie algebra of nilpotent upper triangular matrices.

In this paper, we intended to emphasize the power of the 'geometric approach' (Sections 3–4) that allows to derive algebraic identities about $h_i(x)$'s by modifying, according to certain rules, the corresponding configurations of labelled pseudo-lines. This is why some of our proofs appear to look like just 'See Fig. X' (cf. proofs of Proposition 6.4, Theorem 8.1(i), etc.).

2. The Yang–Baxter equation

Let $\mathscr{A}$ be an associative algebra with identity 1 over a field K of zero characteristic, and let $\{h_i(x) : x \in K,\ i = 1,2,\ldots\}$ be a family of elements of $\mathscr{A}$. (In fact, we will treat x as a formal variable rather than a parameter.) We shall study situations where $h_i(x)$'s satisfy the following conditions:

$$h_i(x)h_j(y) = h_j(y)h_i(x) \quad \text{if } |i-j| \geqslant 2; \tag{2.1}$$

$$h_i(x)h_{i+1}(x+y)h_i(y) = h_{i+1}(y)h_i(x+y)h_{i+1}(x); \tag{2.2}$$

$$h_i(x)h_i(y) = h_i(x+y); \quad h_i(0) = 1. \tag{2.3}$$

The condition (2.2) is one of the forms of the *Yang–Baxter equation* (YBE); (2.3) means that we are interested in *exponential* solutions of the YBE. The most natural way to construct such solutions is the following. Let u_1, u_2, ... be generators of our algebra $\mathscr{A}$; assume they satisfy

$$u_i u_j = u_j u_i, \quad |i-j| \geqslant 2; \tag{2.4}$$

i.e., $\mathscr{A}$ is a *local algebra* in the sense of [30]. Then let

$$h_i(x) = \exp(xu_i); \tag{2.5}$$

we assume that the expression on the right-hand side is well-defined. Then (2.1) and (2.3) are guaranteed and we only need to satisfy the YBE (2.2) which in this case can be rewritten as

$$\exp(xu_i)\exp((x+y)u_{i+1})\exp(yu_i) = \exp(yu_{i+1})\exp((x+y)u_i)\exp(xu_{i+1}). \tag{2.6}$$

Some examples of solutions are given below.

Definition 2.1. A [generalized] *Hecke algebra* (sometimes also called an *Iwahori algebra*) $\mathscr{H}_{a,b}$ is an associative algebra with generators $\{u_i : i = 1, 2, \ldots\}$ satisfying (2.4),

$$u_i u_{i+1} u_i = u_{i+1} u_i u_{i+1}, \tag{2.7}$$

and

$$u_i^2 = au_i + b. \tag{2.8}$$

In particular, $\mathscr{H}_{0,1}$ is the group algebra of the symmetric group.

The corresponding *nilCoxeter algebra* $\mathscr{H}_{0,0}$ (see [14]) defined by (2.4), (2.7), and $u_i^2 = 0$ can be interpreted as the algebra spanned by permutations of S_n, with the multiplication rule

$$w \cdot v = \begin{cases} \text{usual product } wv & \text{if } l(w) + l(v) = l(wv), \\ 0 & \text{otherwise,} \end{cases}$$

where $l(w)$ is the length of a permutation w (the number of inversions).

It is not hard to check that (2.6) holds in $\mathscr{H}_{a,b}$ if $b = 0$. However, we will give an indirect proof of this fact, in order to relate it to some well-known properties of Hecke algebras.

The following statement is implicit in [28].

Lemma 2.2. *Let $c \in K$. The elements $h_i(x) \in \mathscr{H}_{a,b}$ defined by*

$$h_i(x) = 1 + \frac{e^{cx} - 1}{a} u_i \tag{2.9}$$

satisfy (2.1)–(2.2).

Proof. It is convenient to write $[x]$ instead of $(e^{cx}-1)/a$. In this notation, $h_i(x) = 1+[x]u_i$. It is easy to check that $[x+y]=[x]+[y]+a[x][y]$. Now (cf. (2.2))

$$\begin{aligned}&(1+[x]u_i)(1+[x+y]u_{i+1})(1+[y]u_i)\\ &\quad -(1+[y]u_{i+1})(1+[x+y]u_i)(1+[x]u_{i+1})\\ &= ([x]+[y]-[x+y])(u_i-u_{i+1})+[x][y](u_i^2-u_{i+1}^2)\\ &= -\,a[x][y](u_i-u_{i+1})+[x][y](au_i+b-au_{i+1}-b)=0. \qquad \square\end{aligned}$$

Corollary 2.3 (case $a=0$). *The elements $h_i(x)\in\mathscr{H}_{0,b}$ defined by $h_i(x)=1+xu_i$ satisfy* (2.1)–(2.2).

Proof. In (2.9), let $c=a$ and then tend a to 0. $\square$

In the case $a=0$, $b=1$ (the group algebra of the symmetric group) the example of the previous corollary is well-known as the so-called Yang's solution [31] of the Yang–Baxter equation.

Corollary 2.4 (case $b=0$). *Let $c\in K$. The elements $h_i(x)\in\mathscr{H}_{a,0}$ defined by* (2.9) *satisfy* (2.1)–(2.3).

Proof. In this case (2.9) can be rewritten as $h_i(x)=\exp\left(\frac{c}{a}xu_i\right)$, and (2.3) follows. $\square$

In particular, (2.1)–(2.3) hold in the case $a=b=0$ [14, Lemma 3.1]. Thus the elements $h_i(x)=1+xu_i$ of the nilCoxeter algebra of the symmetric group provide an exponential solution of the Yang–Baxter equation. (This can also be easily checked directly.)

3. Geometric interpretation

The relations (2.1)–(2.2) are known to have a nice geometric interpretation (see, e.g., [6]) which is reproduced below; in the next section this interpretation will be modified to involve the condition (2.3) as well.

Suppose we have a family of non-vertical straight lines intersecting a vertical strip on a real plane; no three of these lines meet at the same point. Also assume that an indeterminate is associated with each line. A typical example is presented in Fig. 1. Given such a configuration with n lines, one can define a sequence $s_{a_1}\cdots s_{a_p}$ of adjacent transpositions (a reduced decomposition in the symmetric group S_n) as shown on Fig. 1; in other words, the index a_i of each s_{a_i} indicates which two of adjacent lines (counting bottom-up) get interchanged when we pass the ith intersection point (counting from the left). The product of these generators in the symmetric group corresponds to the permutation defined by a given configuration.

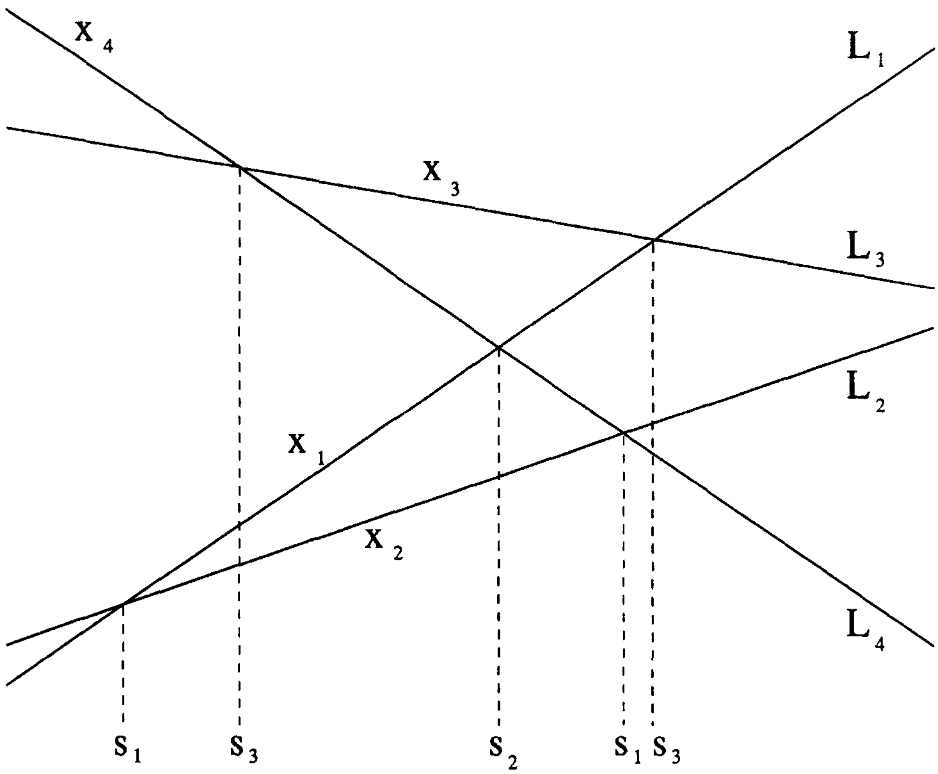

Fig. 1.

Assume conditions (2.1)–(2.2) are satisfied by some elements $\{h_i(x)\}$. Let $\mathscr{C}$ be a configuration of the above-described type. Define

$$\Phi(\mathscr{C};x_1,x_2,\ldots) = h_{a_1}(x_{k_1}-x_{l_1})h_{a_2}(x_{k_2}-x_{l_2})\cdots h_{a_p}(x_{k_p}-x_{l_p}), \tag{3.1}$$

where, as before, $(a_1,\ldots,a_p)$ is a reduced decomposition corresponding to the given configuration, and x_{k_i} and x_{l_i} are the indeterminates for the lines meeting at the ith intersection point; x_{k_i} corresponds to a line with a smaller slope and x_{l_i} to a line with a greater slope.

For example, if $\mathscr{C}$ is the configuration in Fig. 1, then

$$\Phi(\mathscr{C};x_1,x_2,x_3,x_4) = h_1(x_2-x_1)h_3(x_4-x_3)h_2(x_4-x_1)h_1(x_4-x_2)h_3(x_3-x_1).$$

Sometimes, for convenience, we will just write $\Phi(\mathscr{C})$ or $\Phi(x_1,\ldots)$.

Informally, the indeterminate attached to a line can be considered as an angle between this line and, say, the vertical direction (the 'y-axis'); then the difference $x_{k_i}-x_{l_i}$ is an 'angle' corresponding to the ith intersection point.

We are now in a position to interpret conditions (2.1)–(2.2): namely, they mean that those moves of lines which do not change the resulting permutation do not affect the corresponding expression $\Phi(\mathscr{C})$. For example, move line L_4 in Fig. 1 (with x_4 attached)

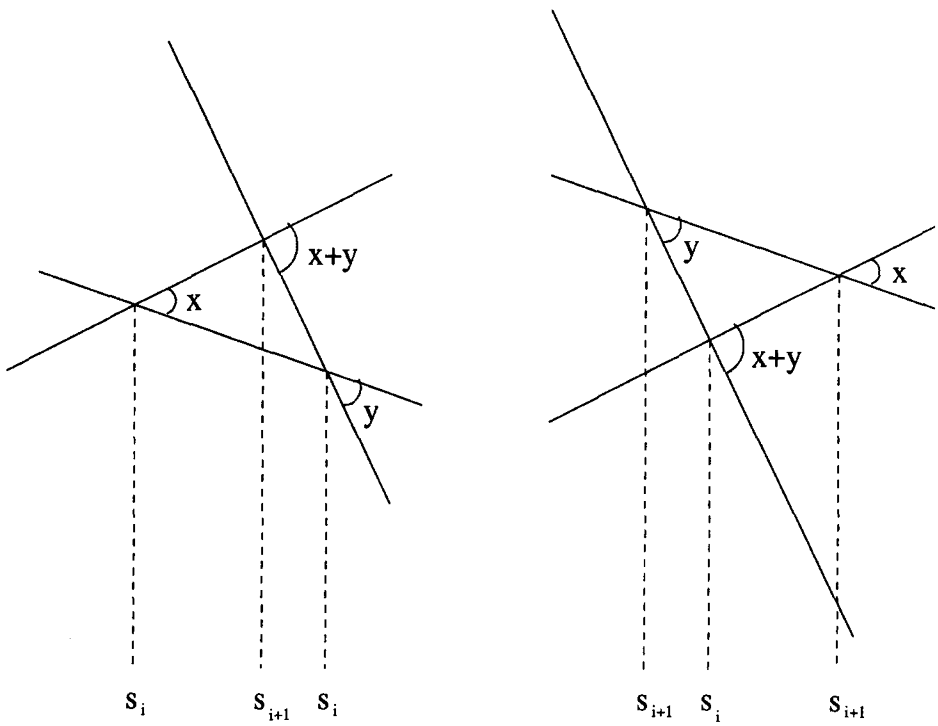

Fig. 2.

a little to the left; then the two leftmost intersection points get interchanged; however, $\Phi(\mathscr{C})$ is left invariant since $h_1(\ldots)$ and $h_3(\ldots)$ commute. Then move L_1 to the right through the intersection point of L_2 and L_4 (be careful that the intersection of L_1 and L_3 does not disappear!). Again, the expression $\Phi(\mathscr{C})$ is invariant because

$$h_1(x_2-x_1)h_2(x_4-x_1)h_1(x_4-x_2) = h_2(x_4-x_2)h_1(x_4-x_1)h_2(x_2-x_1).$$

A general transformation of this type is presented in Fig. 2; it clearly corresponds to (2.2).

The entire construction can be straightforwardly extended to 'pseudo-line configurations'; it means that lines may not be straight, although the following two conditions must hold, as before:

each line is continuous and intersects any vertical line at a single point; (3.2)

any two lines of a configuration have at most one intersection point. (3.3)

4. Generalized configurations

The construction of the previous section can be generalized in the following way. Assume the lines forming a configuration are still continuous but they consist of parts

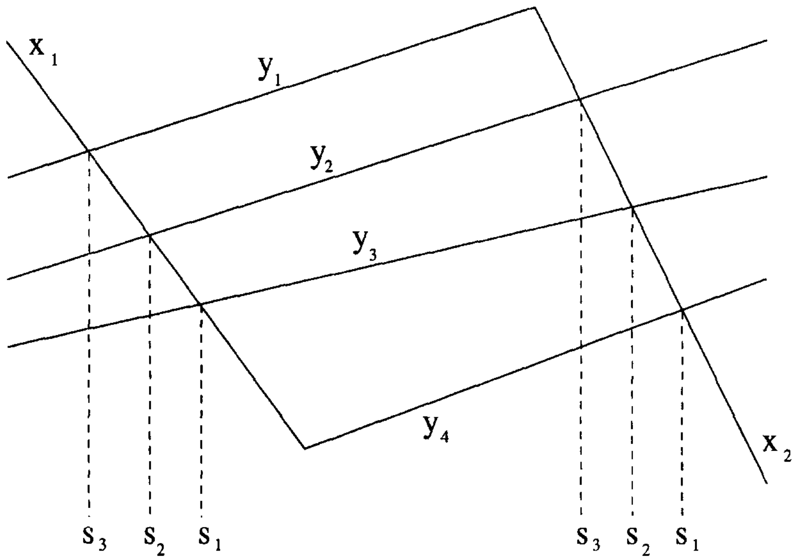

Fig. 3.

(segments); different indeterminates are associated with different segments. A typical configuration of this type appears in Fig. 3 where

$$\Phi(\mathscr{C};x_1,x_2;y_1,\dots,y_4) = h_3(x_1-y_1)h_2(x_1-y_2)h_1(x_1-y_3)h_3(x_2-y_2)$$
$$h_2(x_2-y_3)h_1(x_2-y_4).$$

In a pseudo-line version, (3.3) should be replaced now by the following condition:

$$\text{any two line segments of a configuration have at most one intersection point.} \tag{4.1}$$

Also note that one can define a natural associative operation on the set of generalized configurations with, say, n 'threads' — namely, the glueing. It corresponds to multiplication of respective expressions $\Phi(\mathscr{C})$.

Geometrical interpretation of identities (2.1)–(2.2) remains the same; one should only be careful and *not* move any line through a breakpoint, i.e., through a point separating two segments. (Otherwise the whole expression may change.)

We can also give now an interpretation (or, at least, a consequence) of the condition (2.3) in the language of configurations.

Lemma 4.1. *Assume* (2.1)–(2.3) *are satisfied and a generalized configuration* $\mathscr{C}$ *of* n *lines has a structure shown in Fig. 4. Namely, we mean that all intersection points between the lines marked* $y_2,\dots,y_{n-1}$ *lie inside the quadrangle formed by lines marked* x_1, y_1, x_2, *and* y_n.

Then the expression $\Phi(\mathscr{C})$ *is symmetric in* x_1 *and* x_2.

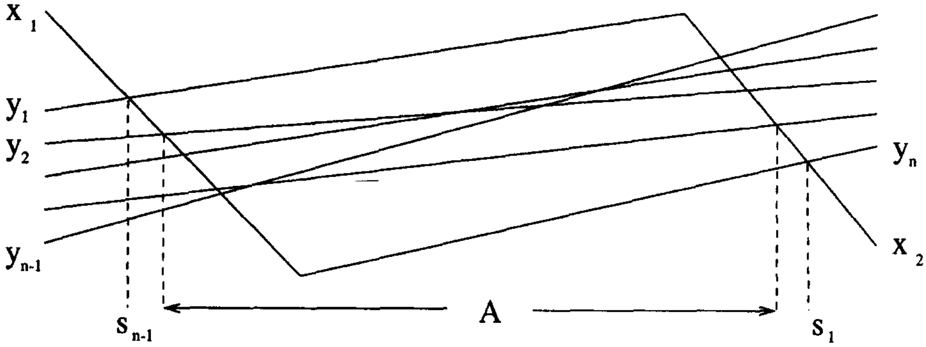

Fig. 4.

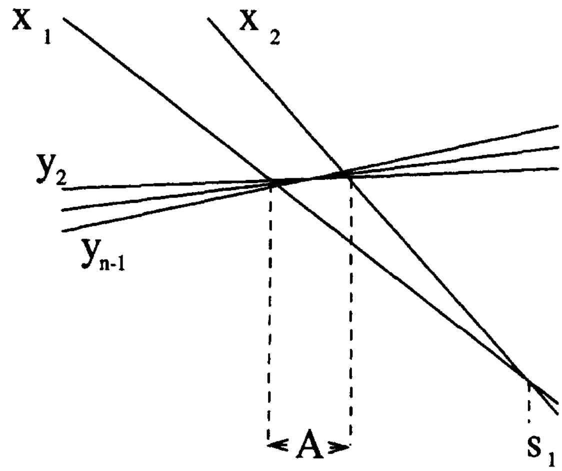

Fig. 5.

Proof. Write

$$\Phi(\mathscr{C}) = h_{n-1}(x_1 - y_1)A(x_1,x_2,y_2,\ldots,y_{n-1})h_1(x_2 - y_n),$$

where $A(\ldots)$ corresponds to 'internal' intersection points (see Fig. 4). The whole expression is claimed to be symmetric in x_1 and x_2. To prove the claim, consider another configuration: remove line segments marked y_1 and y_n and extend lines marked x_1 and x_2 until they intersect. We may assume, without loss of generality, that this new intersection point is on the right-hand side, and no new intersections (among y_i's) appear; see Fig. 5. For the modified configuration $\mathscr{C}'$, one has

$$\Phi(\mathscr{C}') = A(x_1,x_2,y_2,\ldots,y_{n-1})h_1(x_2 - x_1). \tag{4.2}$$

Now move the lines marked x_1 and x_2 so that their intersections with lines corresponding to y_i's get interchanged; the intersection point of our two lines moves to the very left, and so we get

$$\Phi(\mathscr{C}') = h_{n-1}(x_2 - x_1)A(x_2,x_1,y_2,\ldots,y_{n-1}). \tag{4.3}$$

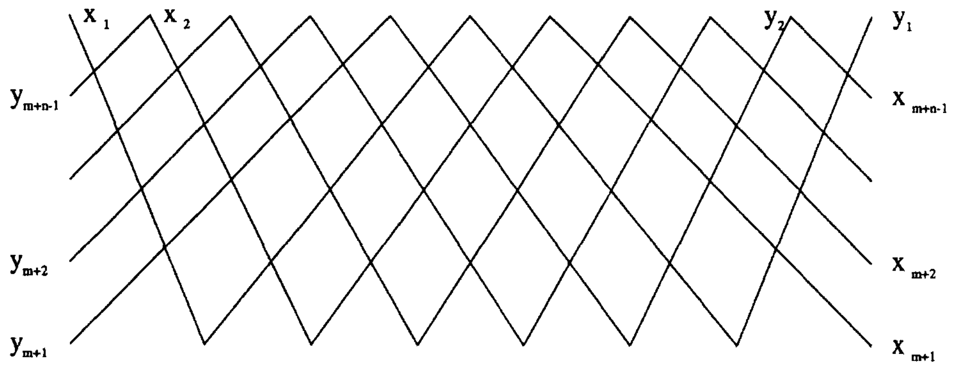

Fig. 6.

Now equate (4.2) and (4.3) and use (2.3) to obtain the claimed identity. □

Note that the whole picture (see Fig. 4) can be reflected in a horizontal line, and the statement of Lemma 4.1 remains valid.

Remark 4.2. Under some natural assumptions, one can also consider *infinite* (to the right, to the left, or both) configurations and define expressions $\Phi(\mathscr{C})$ for them. Namely, let $\Phi(\mathscr{C})$ be the corresponding infinite product of $h_i(x_k - x_l)$'s where $x_1, x_2, \ldots$ are the variables for participating line segments. Assume that each segment of a configuration intersects finitely many other segments. Suppose that $h_i(x)$ is actually some power series in x (this is the case in all our examples). Then $\Phi(\mathscr{C})$ is a power series in x_i's and a computation of a coefficient of each monomial is finite because it only depends on the part of the configuration that contains segments corresponding to participating variables.

5. Symmetric functions

Now we can use Lemma 4.1 to introduce a class of configurations for which the associated expressions are symmetric in many variables.

Corollary 5.1. *Assume conditions* (2.1)–(2.3) *are satisfied. Then the expression*

$$\Phi(\mathscr{C}; x_1, \ldots, x_{m+n-1}; y_1, \ldots, y_{m+n-1})$$

defined by a configuration in Fig. 6 *is symmetric in* $x_1, \ldots, x_{m+1}$ *and, separately, in* $y_1, \ldots, y_{m+1}$.

(Note that it is *not* symmetric in x_i's and y_i's with $i \geqslant m+2$.)

This expression can be formally written as, e.g.,

$$\Phi(\mathscr{C}) = \prod_{d=2-m-n}^{m+n-2} \prod_{\substack{i-j=d \\ m+2 \leqslant i+j \leqslant m+n}} h_{i+j-m-1}(x_i - y_j), \tag{5.1}$$

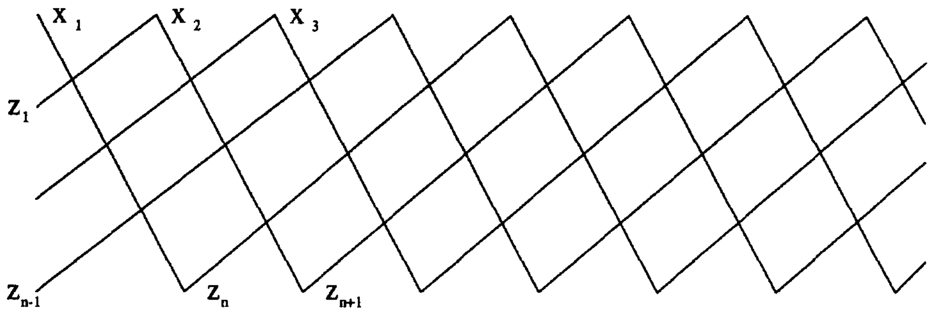

Fig. 7.

where in the first product the factors are multiplied left-to-right, according to the increase of d. (Factors in the second product commute.)

Proof. Follows from Lemma 4.1. □

This corollary has some useful modifications and particular cases. First let us tend m to infinity.

Corollary 5.2. *Assume* (2.1)–(2.3) *hold. Define* $\Phi(\mathscr{C})$ *via an infinite configuration on Fig. 7. Then* $\Phi(\mathscr{C})$ *is symmetric in* $x_1, x_2, \ldots$ *and, separately, in* $z_{n-1}, z_n, z_{n+1}, \ldots$

(Recall Remark 4.2.)

Now we slightly modify the definition of Corollary 5.1/Fig. 6 to make $\Phi(\mathscr{C})$ symmetric in all the x_i's even in the finite setting.

Corollary 5.3. *Assume* (2.1)–(2.3) *hold. Then an expression* $\Phi(\mathscr{C})$ *defined by Fig.* 8 *is symmetric in* $x_1, \ldots, x_{n-1}$.

This expression can be written as

$$\Phi(\mathscr{C}) = \prod_{i=1}^{n-1} \prod_{j=n-1}^{1} h_j(x_i - y_{-i+j+1}),$$

where in both [non-commutative] products the factors are ordered left-to-right as indicated; e.g., the leftmost factor is $h_{n-1}(x_1 - y_{n-1})$ and the rightmost factor is $h_1(x_{n-1} - y_{3-n})$.

The simplest case is one when all the y_i's vanish.

Corollary 5.4. *Let* (2.1)–(2.3) *hold. Define* $A(x) = h_{n-1}(x) \cdots h_2(x) h_1(x)$. *Then, for any* x *and* y, $A(x)$ *and* $A(y)$ *commute. Hence the product*

$$G(x_1, x_2, \ldots) = A(x_1) A(x_2) \cdots \tag{5.2}$$

is symmetric in $x_1, x_2, \ldots$.

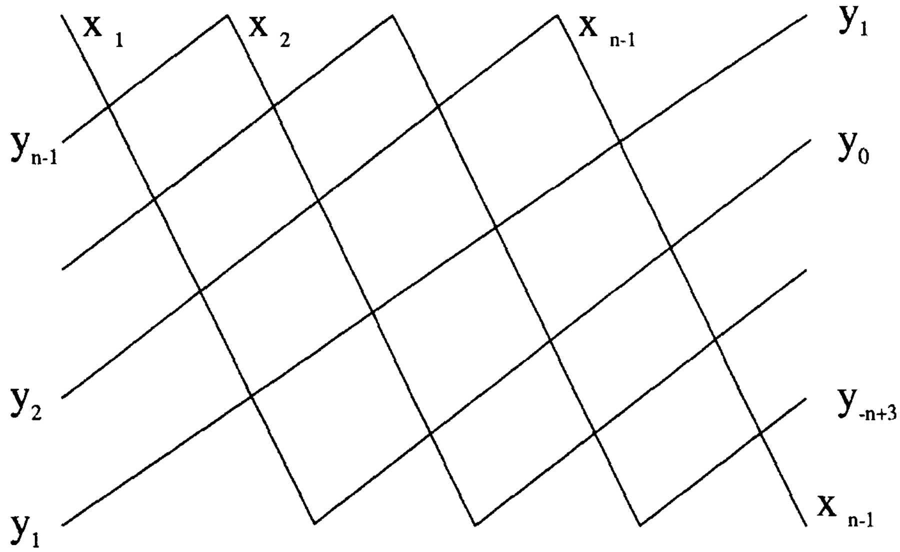

Fig. 8.

This statement generalizes [14, Lemma 2.1].

The above constructions allow us to introduce a whole class of symmetric (or double symmetric) functions in the following way. Take any representation of the algebra $\mathscr{A}$. Apply the operator representing an expression $\Phi(\mathscr{C})$ to an arbitrary vector w; expand the result in an arbitrary linear basis and take any of the coordinates. It will be a symmetric function in the corresponding variables.

The main example is the regular representation. Let W be some linear basis of $\mathscr{A}$. For any $a \in \mathscr{A}$ and $w \in W$ let $\langle a, w\rangle$ denote the respective coordinate of a; in other words, $a = \sum \langle a, w\rangle w$. Now let $\mathscr{C}$ be a (generalized) configuration, and let $w \in W$. Define $\Phi_w(\mathscr{C}) = \langle \Phi(\mathscr{C}), w\rangle$ (cf. [14, (2.3) and below]). The functions Φ_w clearly have (at least) the same symmetry Φ has. Thus the configurations of Figs. 6–8 provide examples of symmetric functions whenever one has found a particular solution of (2.1)–(2.3) and has chosen any basis in the corresponding associativee algebra.

6. Permutations and Schubert polynomials

This section is devoted to studying the simplest solution of Eqs. (2.1)–(2.3), namely, the solution

$$h_i(x) = 1 + xu_i, \tag{6.1}$$

where u_i's are the generators of the nilCoxeter algebra $\mathscr{H}_{0,0}$ (see Section 2). In this case there is a natural basis $W = S_n$ formed by the permutations, and the functions $\Phi_w(\mathscr{C})$ of Section 5 have a nice combinatorial interpretation.

9(a) 9(b)

Fig. 9.

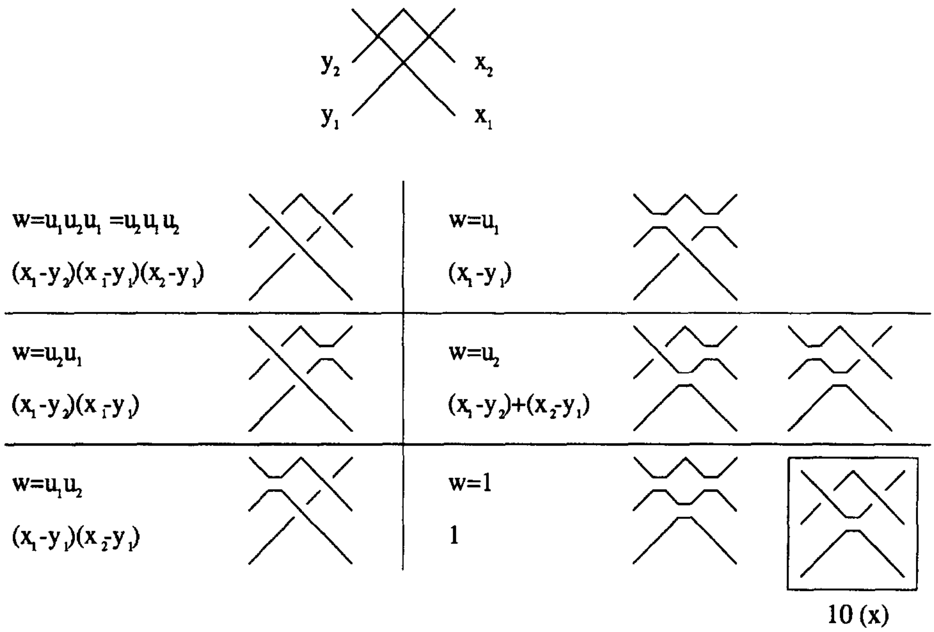

Fig. 10.

Let $\mathscr{C}$ be a generalized configuration (see Section 4), and let $w \in S_n$. One can see directly from the definitions that the function $\Phi_w(\mathscr{C})$ has the following meaning. In the neighborhood of each intersection point, transform the configuration in one of the two ways shown in Fig. 9. (This corresponds to choosing either 1 or $(x-y)u_i$ from the corresponding factor $h_i(x-y) = 1+(x-y)u_i$.) Then we get a *braid* that naturally gives a *permutation*. Now take all the transformations of the initial configuration which lead to the given permutation and satisfy the following condition: *any two threads in the resulting braid intersect at most once*. (This condition ensures we are getting a reduced decomposition, i.e., the corresponding product of generators of the nilCoxeter algebra is the same as it would be in the group algebra of the symmetric group.) For each of these pictures write a product $\prod(x-y)$ computed over all intersection points which were 'resolved' as shown in Fig. 9(b). Then add all these products. The result is $\Phi_w(\mathscr{C})$.

Example 6.1. See Fig. 10. Note that we exclude the picture in Fig. 10 (x) because the upper two braids intersect twice.

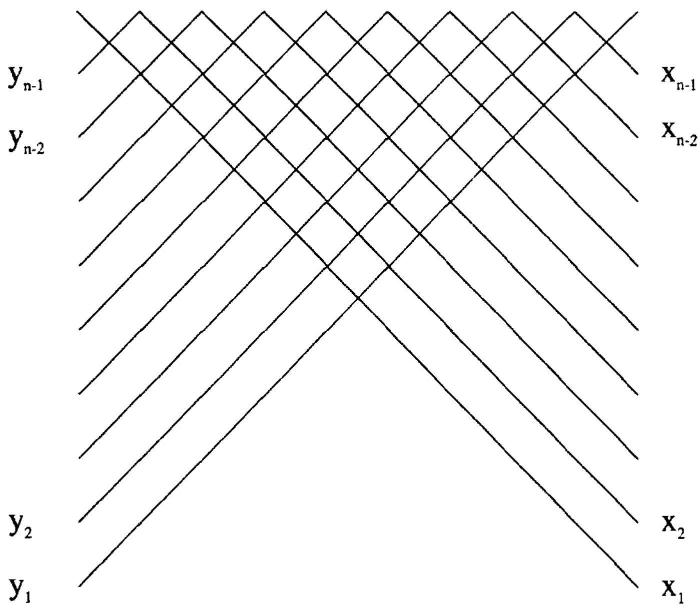

Fig. 11.

Proposition 6.2 ([14]; cf. also [4]). *Let $h_i(x)$ be defined by* (6.1). *Let $\mathscr{C}$ be the configuration in Fig.* 11; *thus*

$$\Phi(\mathscr{C}) = \mathfrak{S}(\boldsymbol{x},\boldsymbol{y}) = \prod_{i=1}^{n-1} \prod_{j=n-i}^{1} h_{i+j-1}(x_i - y_j) \tag{6.2}$$

Then, for any $w \in S_n$, the function $\Phi_w(x_1,\ldots,x_{n-1};-y_1,\ldots,-y_{n-1})$ is the double Schubert polynomial of Lascoux and Schützenberger.

See, e.g., [25,22] for the usual definition of the Schubert polynomials via divided differences. These polynomials are usually denoted $\mathfrak{S}_w$; we will also use this notation (cf. Section 8).

In particular, for $y_1 = y_2 = \cdots = 0$ we get ordinary Schubert polynomials [23,3,8,25,22]. Thus Example 6.1 gives a computation of all Schubert polynomials for the symmetric group S_3.

Note that the configuration in Fig. 11 is a special case $m = 0$ of the one in Fig. 6.

Proposition 6.3. *Assume, as before, that $h_i(x)$'s are defined by* (6.1). *Then, for $w \in S_n$, the function G_w defined by* (5.2) *is the so-called stable Schubert polynomial or Stanley's symmetric function.*

See [14] or [4] for a definition of G_w which essentially coincides with that of ours. The original definition appeared in [29]; see also [23]. Kraśkiewicz and

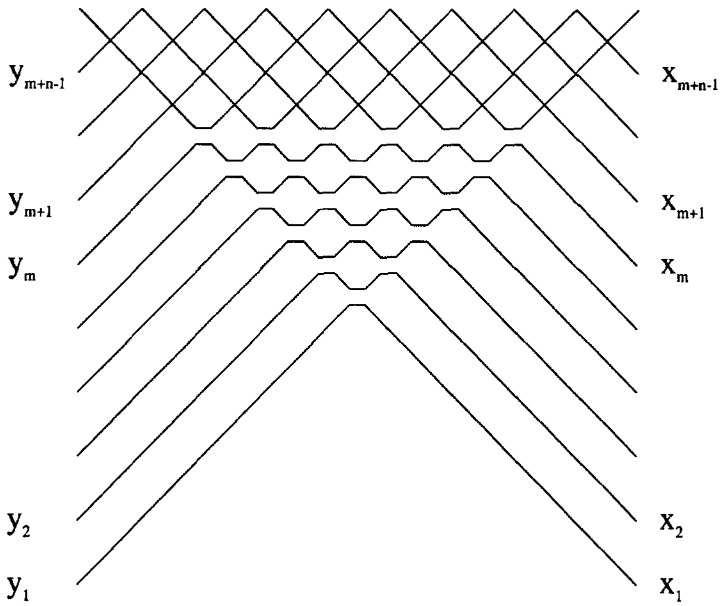

Fig. 12.

Pragacz [19] constructed representations of S_n which correspond to G_w's; see also [18]

Sometimes it is more natural and convenient to work with respective symmetric functions in infinitely many variables. To do this, take the configuration in Fig. 7 and set $z_i = 0$ for all i's. It results in a Stanley's symmetric function in infinitely many variables $x_1, x_2, \ldots$ One can also consider more general 'double Stanley polynomials' (or 'double stable Schubert polynomials') $G_w(x_1, x_2, \ldots; z_1, z_2, \ldots)$ which are symmetric in x_i's and, separately, in z_i's for $i \geqslant n-1$.

We are going to clarify now why the G_w's are called the stable Schubert polynomials.

Let $w \in S_n$ be a permutation regarded as a bijection $\{1, \ldots, n\} \to \{1, \ldots, n\}$, and m a positive integer. Define a permutation $1_m \times w \in S_{n+m}$ by

$$(1_m \times w)(i) = \begin{cases} i & \text{if } i \leqslant m, \\ m + w(i-m) & \text{if } i > m. \end{cases}$$

In other notation, if $w = w_1 \ldots w_n$, then $1_m \times w = 12 \ldots m(m+w_1) \ldots (m+w_n)$.

Proposition 6.4. *Let $w \in S_n$. Then the double Schubert polynomial*

$$\mathfrak{S}_{1_m \times w}(x_1, \ldots, x_{m+n-1}; -y_1, \ldots, -y_{m+n-1})$$

coincides with the polynomial $\Phi_w(\mathscr{C})$ where $\mathscr{C}$ is the configuration in Fig. 6.

Proof. Look at Fig. 12. □

Now we can tend m to infinity and get, as a limiting case, the configuration in Fig. 7 which corresponds to the double Stanley polynomial in infinitely many variables. Thus we obtain a 'super-symmetric version' of the well-known result [23,4,14]: Stanley's polynomials are the stable Schubert polynomials.

Corollary 6.5. *For any permutation w, $\lim_{m\to\infty} \mathfrak{S}_{1_m\times w} = G_w$ where the limit means that the coefficient of each particular monomial in the expansion of $\mathfrak{S}_{1_m\times w}$ gets fixed when m is sufficiently large.*

7. Enveloping algebra of $U_+(gl(n))$

Let $\mathscr{A}$ be the universal enveloping algebra of the Lie algebra of the upper triangular matrices with zero main diagonal. Then $\mathscr{A}$ can be defined as generated by u_1, u_2, ... satisfying (2.4) and the Serre relations

$$[u_i,[u_i,u_{i\pm 1}]] = 0, \tag{7.1}$$

where [,] stands for commutator: $[a,b] = ab - ba$. We will show that this algebra provides another example of an exponential solution of the Yang–Baxter equation. In other words, (2.6) holds; thus the elements $h_i(x) = \exp(xu_i)$ satisfy (2.1)–(2.3). Hence one can define corresponding symmetric functions as well as certain analogues of the Schubert polynomials related to this specific solution.

Theorem 7.1. *Relations* (2.4) *and* (7.1) *imply* (2.6).

Proof. Let us redenote $a = u_i$, $b = u_{i+1}$. So we need to prove that $[a,[a,b]] = [b,[a,b]] = 0$ implies $[\exp(xa)\exp(xb), \exp(yb)\exp(ya)] = 0$.

It suffices to show that the coefficient T_n of $x^n/n!$ in $\exp(xa)\exp(xb)$ commutes with the coefficient S_m of $y^m/m!$ in $\exp(yb)\exp(ya)$. Let $\mathscr{L}$ be the algebra generated by $a+b$ and $[a,b]$. We will prove that $T_n \in \mathscr{L}$. Then, similarly, $S_m \in \mathscr{L}$ and they commute because $\mathscr{L}$ is commutative. Now note that $T_n = \sum\binom{n}{k}a^k b^{n-k}$ and therefore $T_{n+1} = aT_n + T_n b$. So our claim follows from the following lemma.

Lemma 7.2. *If $T \in \mathscr{L}$, then $aT + Tb \in \mathscr{L}$.*

Proof. Since $aT + Tb = (a+b)T + [T,b]$, we need to prove that $[T,b] \in \mathscr{L}$. We can assume that T is a monomial in $a+b$ and $[a,b]$. Now take Tb and move b to the left through all the factors; each of these is either $(a+b)$ or $[a,b]$. While moving, we will be getting in each step an additional term which is either $[a+b,b]$ or $[[a,b],b]$ surrounded by expressions belonging to $\mathscr{L}$. Since both $[a+b,b] \in \mathscr{L}$ and $[[a,b],b] \in \mathscr{L}$, this completes the proof of Lemma 7.2 and Theorem 7.1. □

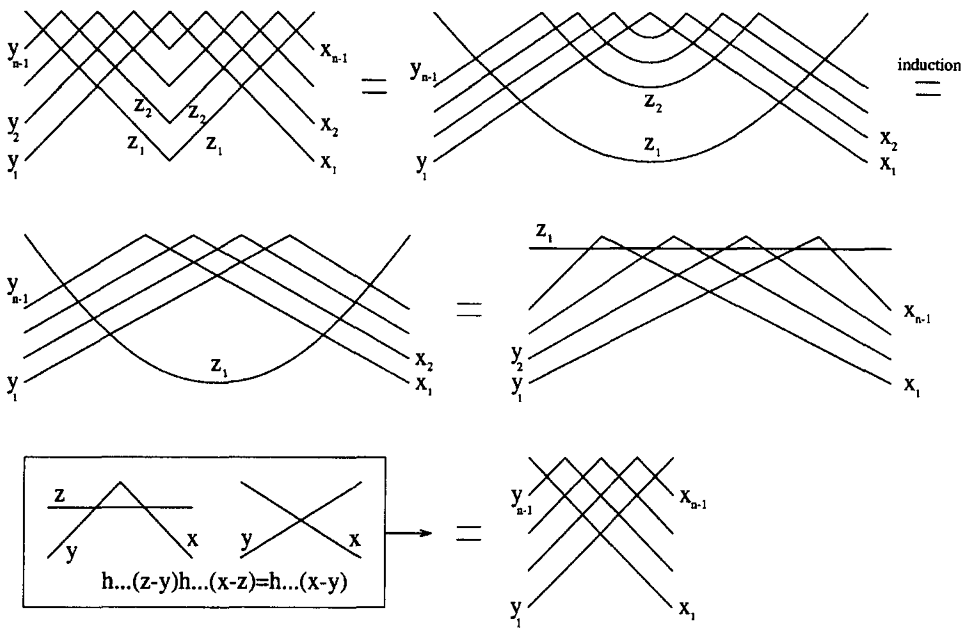

Fig. 13.

8. Cauchy-type identities

Let $\mathfrak{S}(\boldsymbol{x},\boldsymbol{y}) = \mathfrak{S}(x_1,\ldots,x_{n-1};y_1,\ldots,y_{n-1})$ denote the generalized double Schubert expression; in other words, $\mathfrak{S}(\boldsymbol{x},\boldsymbol{y}) = \Phi(\mathscr{C})$ where $\mathscr{C}$ is as shown in Fig. 11.

Theorem 8.1. (i) $\mathfrak{S}(\boldsymbol{z},\boldsymbol{y})\mathfrak{S}(\boldsymbol{x},\boldsymbol{z}) = \mathfrak{S}(\boldsymbol{x},\boldsymbol{y})$.
(ii) $\mathfrak{S}(\boldsymbol{x},\boldsymbol{x}) = 1$.

Proof. (i) See Fig. 13. (ii) Let $\boldsymbol{x} = \boldsymbol{y} = \boldsymbol{0}$; then (i) gives $1 = \mathfrak{S}(\boldsymbol{0},\boldsymbol{0}) = \mathfrak{S}(\boldsymbol{z},\boldsymbol{0})\mathfrak{S}(\boldsymbol{0},\boldsymbol{z})$ which implies that $1 = \mathfrak{S}(\boldsymbol{0},\boldsymbol{z})\mathfrak{S}(\boldsymbol{z},\boldsymbol{0}) = \mathfrak{S}(\boldsymbol{z},\boldsymbol{z})$, as desired. □

Theorem 8.1(i) generalizes [14, Lemma 4.5] and [25, pp. 87–88]. (Our proof is essentially a modified geometric version of the proof in [14].) In the nilCoxeter case, it tells (after the substitution $\boldsymbol{y} \longleftarrow -\boldsymbol{y}$) that

$$\mathfrak{S}_w(\boldsymbol{x},\boldsymbol{y}) = \sum_{\substack{uv=w\\ l(u)+l(v)=l(w)}} \mathfrak{S}_u(\boldsymbol{z},\boldsymbol{y})\mathfrak{S}_v(\boldsymbol{x},-\boldsymbol{z}).$$

When $\boldsymbol{z} = \boldsymbol{0} = (0,\ldots,0)$, Theorem 8.1(i) reduces to $\mathfrak{S}(\boldsymbol{0},\boldsymbol{y})\mathfrak{S}(\boldsymbol{x},\boldsymbol{0}) = \mathfrak{S}(\boldsymbol{x},\boldsymbol{y})$, a formula that allows to express generalized double Schubert polynomials in terms of 'ordinary' ones (i.e., not double but still generalized); cf. [21,25,14]. Note that in the nilCoxeter case $\mathfrak{S}_w(\boldsymbol{x},\boldsymbol{0}) = \mathfrak{S}_w(\boldsymbol{x})$ and $\mathfrak{S}_w(\boldsymbol{0},\boldsymbol{y}) = \mathfrak{S}_{w^{-1}}(-\boldsymbol{y})$.

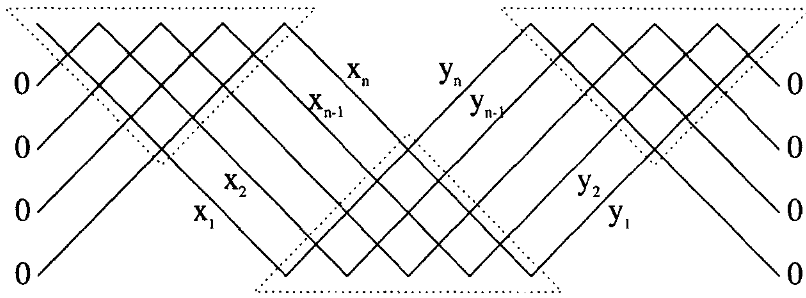

Fig. 14.

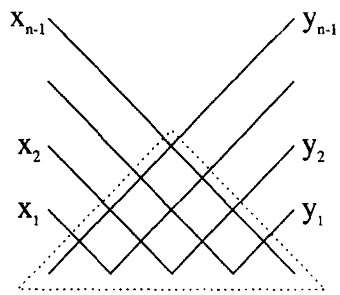

Fig. 15.

Let G denote the expression $\Phi(\mathscr{C})$ defined by the configuration $\mathscr{C}$ in Fig. 6 with $m = n$ and $x_{n+1} = x_{n+2} = \cdots = y_{n+1} = y_{n+2} = \cdots = 0$ (see Fig. 14). This is a generalized 'supersymmetric Stanley expression' in the variables $x_1, \ldots, x_n$, $y_1, \ldots, y_n$.

Theorem 8.2. *Let*

$$\check{\mathfrak{S}}\,(\boldsymbol{x},\boldsymbol{y}) = \prod_{i=1}^{n-1} \prod_{j=i}^{1} h_j(x_i - y_{n-i+j-1})$$

be the 'flipped' Schubert expression; see Fig. 15. (*Do not confuse* $\check{\mathfrak{S}}$ *with* $\tilde{\mathfrak{S}}$ *of* [14].) *Denote, as before,* $\boldsymbol{x} = (x_1, \ldots, x_{n-1})$ *and* $\boldsymbol{y} = (y_1, \ldots, y_{n-1})$. *Then*

$$G(x_1, \ldots, x_n; y_1, \ldots, y_n) = \mathfrak{S}\,(\boldsymbol{x},\boldsymbol{0})\check{\mathfrak{S}}\,(x_2, \ldots, x_n; y_2, \ldots, y_n)\mathfrak{S}\,(\boldsymbol{0},\boldsymbol{y}).$$

Proof. See Fig. 14; configurations are identified with corresponding expressions. □

In the nilCoxeter case,

$$\check{\mathfrak{S}}\,(\boldsymbol{x},\boldsymbol{y}) = \sum_{w} \mathfrak{S}_w(x_{n-1}, \ldots, x_1; y_{n-1}, \ldots, y_1)\; w_0 w^{-1} w_0;$$

this follows from the fact that we can obtain Fig. 15 by first flipping it in a vertical line, then flipping it upside down, and then renumbering x_i's and y_j's the other way around.

Corollary 8.3. *For the double Stanley polynomials* $G_w = G_w(x_1, \ldots, x_n; y_1, \ldots, y_n)$,

$$G_w = \sum_{\substack{uvp=w \\ l(u)+l(v)+l(p)=l(w)}} \mathfrak{S}_u(\boldsymbol{x})\mathfrak{S}_{w_0v^{-1}w_0}(x_n, \ldots, x_2; y_n, \ldots, y_2)\mathfrak{S}_{p^{-1}}(-\boldsymbol{y}) \ .$$

Setting $y_1 = y_2 = \cdots = 0$, we obtain an exact expression for Stanley's polynomials in terms of Schubert's. Note that G_w, being a homogeneous symmetric function that expands into a sum of Schur functions whose shapes have at most $n-1$ columns (see [10]), is uniquely defined by its n-variables specialization.

Corollary 8.4.

$$G_w(x_1, \ldots, x_n) = \sum_{\substack{uv=w \\ l(u)+l(v)=l(w)}} \mathfrak{S}_u(x_1, \ldots, x_{n-1})\mathfrak{S}_{w_0v^{-1}w_0}(x_n, \ldots, x_2).$$

Theorem 8.5.

$$G(x_1, \ldots, x_n; z_1, \ldots, z_n)G(z_1, \ldots, z_n; y_1, \ldots, y_n) = G(x_1, \ldots, x_n; y_1, \ldots, y_n).$$

Proof 1 (*nilCoxeter case only*). Derive from Theorem 8.1(i) and Proposition 6.4. □

Proof 2. By analogy with Theorem 8.1(i), $\check{\mathfrak{S}}(\boldsymbol{x},\boldsymbol{z})\check{\mathfrak{S}}(\boldsymbol{z},\boldsymbol{y}) = \check{\mathfrak{S}}(\boldsymbol{x},\boldsymbol{y})$. Use this observation and Theorem 8.2 to obtain

$$\begin{aligned}
&G(x_1, \ldots, x_n; z_1, \ldots, z_n)G(z_1, \ldots, z_n; y_1, \ldots, y_n) \\
&\quad = \mathfrak{S}(\boldsymbol{x},\boldsymbol{0})\check{\mathfrak{S}}(x_2, \ldots, x_n; z_2, \ldots, z_n)\mathfrak{S}(\boldsymbol{0},\boldsymbol{z})\mathfrak{S}(\boldsymbol{z},\boldsymbol{0}) \\
&\qquad \times \check{\mathfrak{S}}(z_2, \ldots, z_n; y_2, \ldots, y_n)\mathfrak{S}(\boldsymbol{0},\boldsymbol{y}) \\
&\quad = \mathfrak{S}(\boldsymbol{x},\boldsymbol{0})\check{\mathfrak{S}}(x_2, \ldots, x_n; z_2, \ldots, z_n)\check{\mathfrak{S}}(z_2, \ldots, z_n; y_2, \ldots, y_n)\mathfrak{S}(\boldsymbol{0},\boldsymbol{y}) \\
&\quad = \mathfrak{S}(\boldsymbol{x},\boldsymbol{0})\check{\mathfrak{S}}(x_2, \ldots, x_n; y_2, \ldots, y_n)\mathfrak{S}(\boldsymbol{0},\boldsymbol{y}) = G(x_1, \ldots, x_n; y_1, \ldots, y_n).
\end{aligned}$$ □

Corollary 8.6.

$$G(x_1, \ldots, x_n; x_1, \ldots, x_n) = 1.$$

Proof. Same reasoning as in the proof of Theorem 8.1(ii). □

Corollary 8.7.

(i) $$G_w(\boldsymbol{x},\boldsymbol{y}) = \sum_{\substack{uv=w \\ l(u)+l(v)=l(w)}} G_u(\boldsymbol{x},\boldsymbol{z})G_v(\boldsymbol{z},\boldsymbol{y});$$

(ii) $$G_w(\boldsymbol{x},\boldsymbol{y}) = \sum_{\substack{uv=w \\ l(u)+l(v)=l(w)}} G_u(\boldsymbol{x})G_{v^{-1}}(-\boldsymbol{y}).$$ □

The last identity has the following interpretation. One can see that the canonical involution ω of the space of symmetric functions (see [24]) sends G_v to $G_{v^{-1}}$. On the other hand, the definition of G_w's (see Fig. 6 or 14) implies that

$$G_w(x_1,\ldots,x_n,y_1,\ldots,y_n;\,0,\ldots,0) = \sum_{\substack{uv=w\\ l(u)+l(v)=l(w)}} G_u(x_1,\ldots,x_n)G_v(y_1,\ldots,y_n).$$

Applying ω to the y_j's only, we obtain a formula for the superfication of G_w's:

$$G_w^{\text{super}}(x_1,\ldots;y_1,\ldots) = \sum_{\substack{uv=w\\ l(u)+l(v)=l(w)}} G_u(x_1,\ldots)G_{v^{-1}}(y_1,\ldots) = G_w(x_1,\ldots;-y_1,\ldots).$$

In other words, $G_w(\boldsymbol{x},-\boldsymbol{y})$ is the canonical superfication of $G_w(\boldsymbol{x})$. In the case when w is a 321-avoiding permutation (see, e.g., [4]), this statement reduces to the recently found new formula for the [skew] super-Schur functions [15,26].

9. Specializations

In this section some computations made in [14] are generalized and simplified. First we treat the special case when $x_1 = x_2 = \cdots$, $y_1 = y_2 = \cdots$.

Lemma 9.1. *Let* $\boldsymbol{c} = (c,c,\ldots)$ *where* $c \in K$. *Then* $\mathfrak{S}(\boldsymbol{x}+\boldsymbol{c},\boldsymbol{y}+\boldsymbol{c}) = \mathfrak{S}(\boldsymbol{x},\boldsymbol{y})$. *The same is true for* $\check{\mathfrak{S}}$ *and* G.

Proof. $\prod h_{\ldots}((x_i+c)-(y_j+c)) = \prod h_{\ldots}(x_i-y_j)$. □

Lemma 9.2 (cf. [14, Lemma 5.1; 25, p. 89]). *Let* $\boldsymbol{x} = (x,x,\ldots)$, $\boldsymbol{y} = (y,y,\ldots)$. *Then* $\mathfrak{S}(\boldsymbol{x})\mathfrak{S}(\boldsymbol{y}) = \mathfrak{S}(\boldsymbol{x}+\boldsymbol{y})$.

Proof. Lemma 9.1 and Theorem 8.1(i) imply

$$\begin{aligned}\mathfrak{S}(\boldsymbol{x}+\boldsymbol{y}) &= \mathfrak{S}(\boldsymbol{x}+\boldsymbol{y},\boldsymbol{0}) = \mathfrak{S}(\boldsymbol{y},-\boldsymbol{x}) = \mathfrak{S}(\boldsymbol{0},-\boldsymbol{x})\mathfrak{S}(\boldsymbol{y},\boldsymbol{0})\\ &= \mathfrak{S}(\boldsymbol{x},\boldsymbol{0})\mathfrak{S}(\boldsymbol{y},\boldsymbol{0}). \qquad \square\end{aligned}$$

Theorem 9.3 (cf. [14, Lemma 2.3; 25, (6.11)]). *Assume* (2.4)-(2.6) *hold. Then*

$$\mathfrak{S}(x,x,\ldots) = \exp(x\cdot(u_1+2u_2+3u_3+\ldots)).$$

Proof. Coincides with the proof of [14, Lemma 2.3]. □

Let us return now to the general case.

Theorem 9.4. *Let x_1, x_2, ... be an infinite sequence of formal variables. Then*

$$\mathfrak{S}(x_1,\ldots,x_{n-1}) = \prod_{k=\infty}^{1} \prod_{j=n-1}^{1} h_j(x_k - x_{k+j}),$$

where in the [non-commutative] products the factors are multiplied in decreasing order (with respect to k and j).

Proof. Use a pictorial representation and Corollary 8.6 to see that

$$\prod_{j=n-1}^{1} h_j(x_k - x_{k+j})\mathfrak{S}(\mathbf{0},\boldsymbol{x}) = G(\ldots,x_2,x_1,0,0,\ldots;\ \ldots,x_2,x_1,0,0,\ldots) = 1;$$

then it only remains to recall that $\mathfrak{S}(\mathbf{0},\boldsymbol{x}) = (\mathfrak{S}(\boldsymbol{x},\mathbf{0}))^{-1}$. □

Corollary 9.5 (14, Lemma 5.3). $\mathfrak{S}(1,q,\ldots,q^{n-2}) = \prod_{i=\infty}^{0}\prod_{j=n-1}^{1} h_j(q^j - q^{i+j})$.

As shown in [14, Theorem 2.4] Corollary 9.5 can be used to obtain an explicit formula for the principal specialization of a Schubert polynomial (conjectured in [25, $(6.11_q?)$]).

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Added in press

The approach presented in this paper was then used by the authors to construct the B_n-analogues of the Schubert polynomials [13] and give the first combinatorial interpretation of the Grothendieck polynomials of Lascoux and Schützenberger [12]. We also described explicitly [11] the universal solution of the basic commutation relations (2.1)-(2.3).

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